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Lectures on Credit Risk

1. Models for single default
2. **Contagion models**
3. Credit derivatives

Contagion Models

1. Toy Model
2. Copula based approach
3. Density Model
4. Markov-Chains
5. Self-exciting Models, Multiphase models
6. Filtering

Toy Model

We assume that

- two default times are given: $\tau_i, i = 1, 2$
- $H_t^i = 1\!\!1_{\tau_i \le t}$ are the default processes,
- $\mathbb{H}^i$ is the natural filtration of H^i,
- $\mathbb{H}$ is the filtration

$$\mathcal{H}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2$$

We introduce the *joint survival process* $G(u,v)$: for every $u,v \in \mathbb{R}_+$,

$$G(u,v) = \mathbb{P}(\tau_1 > u, \tau_2 > v)$$

We write

$$\partial_1 G(u,v) = \frac{\partial G}{\partial u}(u,v), \quad \partial_{12} G(u,v) = \frac{\partial^2 G}{\partial u \partial v}(u,v).$$

We assume that the joint density $f(u,v) = \partial_{12} G(u,v)$ exists. In other words, we postulate that $G(u,v)$ can be represented as follows

$$G(u,v) = \int_u^\infty \Big(\int_v^\infty f(x,y)\, dy \Big)\, dx.$$

We compute conditional expectation in the filtration $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$ from the Key lemma with $\mathbb{F} = \mathbb{H}^2$, $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$, $X = 1$ and $\tau = \tau_1$:

For $t < T$

$$\mathbb{E}(X \mathbb{1}_{T<\tau} | \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \frac{\mathbb{E}(X \mathbb{1}_{\{\tau > T\}} | \mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau > t\}} | \mathcal{F}_t)}$$

$$\mathbb{P}(T < \tau_1 | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) = \mathbb{1}_{\{t<\tau_1\}} \frac{\mathbb{P}(T < \tau_1 | \mathcal{H}_t^2)}{\mathbb{P}(t < \tau_1 | \mathcal{H}_t^2)}$$

$$= \mathbb{1}_{t<\tau_1} \left(\mathbb{1}_{t<\tau_2} \frac{\mathbb{P}(T < \tau_1, t < \tau_2)}{\mathbb{P}(t < \tau_1, t < \tau_2)} + \mathbb{1}_{\tau_2 \le t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right)$$

$$= \mathbb{1}_{t<\tau_1} \left(\mathbb{1}_{t<\tau_2} \frac{G(T,t)}{G(t,t)} + \mathbb{1}_{\tau_2 \le t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right)$$

We compute conditional expectation in the filtration $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$ from the Key lemma with $\mathbb{F} = \mathbb{H}^2$, $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$, $X = 1$ and $\tau = \tau_1$:
For $t < T$

$$\begin{aligned}
\mathbb{E}(X \mathbb{1}_{T<\tau} | \mathcal{G}_t) &= \mathbb{1}_{\{t<\tau\}} \frac{\mathbb{E}(X \mathbb{1}_{\{\tau>T\}} | \mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau>t\}} | \mathcal{F}_t)} \\
\mathbb{P}(T < \tau_1 | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) &= \mathbb{1}_{t<\tau_1} \frac{\mathbb{P}(T < \tau_1 | \mathcal{H}_t^2)}{\mathbb{P}(t < \tau_1 | \mathcal{H}_t^2)}
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{1}_{t<\tau_1} \left(\mathbb{1}_{t<\tau_2} \frac{\mathbb{P}(T < \tau_1, t < \tau_2)}{\mathbb{P}(t < \tau_1, t < \tau_2)} + \mathbb{1}_{\tau_2 \le t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) \\
&= \mathbb{1}_{t<\tau_1} \left(\mathbb{1}_{t<\tau_2} \frac{G(T,t)}{G(t,t)} + \mathbb{1}_{\tau_2 \le t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right)
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We compute conditional expectation in the filtration $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$ from the Key lemma with $\mathbb{F} = \mathbb{H}^2$, $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$, $X = 1$ and $\tau = \tau_1$:
For $t < T$

$$\begin{aligned} \mathbb{E}(X 1\!\!1_{T<\tau} | \mathcal{G}_t) &= 1\!\!1_{\{\tau > t\}} \frac{\mathbb{E}(X 1\!\!1_{\{\tau > T\}} | \mathcal{F}_t)}{\mathbb{E}(1\!\!1_{\{\tau > t\}} | \mathcal{F}_t)} \\ \mathbb{P}(T < \tau_1 | \mathcal{H}^1_t \vee \mathcal{H}^2_t) &= 1\!\!1_{t<\tau_1} \frac{\mathbb{P}(T < \tau_1 | \mathcal{H}^2_t)}{\mathbb{P}(t < \tau_1 | \mathcal{H}^2_t)} \end{aligned}$$

A second application of the key lemma leads to

$$\begin{aligned} &= 1\!\!1_{\{t<\tau_1\}} \left(1\!\!1_{\{t<\tau_2\}} \frac{\mathbb{P}(T < \tau_1, t < \tau_2)}{\mathbb{P}(t < \tau_1, t < \tau_2)} + 1\!\!1_{\{\tau_2 \leq t_=} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) \\ &= 1\!\!1_{\{t<\tau_1\}} \left(1\!\!1_{\{t<\tau_2\}} \frac{G(T,t)}{G(t,t)} + 1\!\!1_{\{\tau_2 \leq t\}} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) \end{aligned}$$

- The computation of $\mathbb{P}(T < \tau_1 | \tau_2)$ can be done as follows:

$$\mathbb{P}(T < \tau_1 | \tau_2 = v) = \frac{\mathbb{P}(T < \tau_1, \tau_2 \in dv)}{\mathbb{P}(\tau_2 \in dv)} = \frac{\partial_2 G(T, v)}{\partial_2 G(0, v)}$$

hence, on the set $\tau_2 < T$,

$$\mathbb{P}(T < \tau_1 | \tau_2) = \frac{\partial_2 G(T, \tau_2)}{\partial_2 G(0, \tau_2)}$$

Value of credit derivatives

We introduce different credit derivatives

A **defaultable zero-coupon** related to the default times τ_i delivers 1 monetary unit if τ_i is greater that T: $D^i(t,T) = \mathbb{E}(1\!\!1_{\{T<\tau_i\}} | \mathcal{H}_t^1 \vee \mathcal{H}_t^2)$

We obtain

$$D^1(t,T) \quad = \quad 1\!\!1_{\{\tau_1>t\}} \left(1\!\!1_{\{\tau_2 \le t\}} \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} + 1\!\!1_{\{\tau_2>t\}} \frac{G(T,t)}{G(t,t)} \right)$$

Some easy computation leads to

$$\mathbb{E}(h(\tau_1,\tau_2)|\mathcal{H}_t) = I_t(1,1)h(\tau_1,\tau_2) + I_t(1,0)\Psi_{1,0}(\tau_1) + I_t(0,1)\Psi_{0,1}(\tau_2) + I_t(0,0)\Psi_{0,0}$$

where

$$\begin{aligned}
\Psi_{1,0}(u) &= -\frac{1}{\partial_1 G(u,t)}\int_t^\infty h(u,v)\partial_1 G(u,dv)\\
\Psi_{0,1}(v) &= -\frac{1}{\partial_2 G(t,v)}\int_t^\infty h(u,v)\partial_2 G(du,v)\\
\Psi_{0,0} &= \frac{1}{G(t,t)}\int_t^\infty\int_t^\infty h(u,v)G(du,dv)
\end{aligned}$$

$$I_t(1,1) = \mathbb{1}_{\{\tau_1\le t,\tau_2\le t\}}\ , \qquad I_t(0,0) = \mathbb{1}_{\{\tau_1> t,\tau_2> t\}}$$

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Intensities

The process

$$H_t^1 - \int_0^{t\wedge\tau_1\wedge\tau_2} \widetilde{\lambda}_u^1\, du - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \lambda^{1|2}(u,\tau_2)\, du,$$

is a $\mathbb{G}$-martingale where

$$\widetilde{\lambda}_t^1 = -\frac{\partial_1 G(t,t)}{G(t,t)}, \quad \lambda^{1|2}(t,s) = -\frac{f(t,s)}{\partial_2 G(t,s)}$$

Note that

$$\begin{aligned}
\widetilde{\lambda}_t^1 &= \lim_{h\to 0}\frac{1}{h}\frac{\mathbb{P}(t<\tau_1\leq t+h,\tau_2>t)}{\mathbb{P}(\tau_1>t,\tau_2>t)} = -\frac{\partial_1 G(t,t)}{G(t,t)} \\
\lambda^{1|2}(t,s) &= \lim_{h\to 0}\frac{1}{h}\mathbb{P}(\tau_1\in[t,t+h]|\tau_2) = -\frac{f(t,\tau_2)}{\partial_2 G(t,\tau_2)}
\end{aligned}$$

From

$$\mathbb{P}(\tau_1 > s|\mathcal{H}_t^2) = (1 - H_t^2)\frac{G(s,t)}{G(0,t)} + H_t^2\frac{\partial_2 G(s,\tau_2)}{\partial_2 G(0,\tau_2)}$$

we deduce that

$$G_t = \mathbb{P}(\tau_1 > t|\mathcal{H}_t^2)$$

$\mathbb{P}(\tau_1 > t|\mathcal{H}_t^2)$ admits a Doob-Meyer decomposition as

$$dG_t = \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right) dM_t^2 + \left(H_t^2\frac{\partial_{1,2} G(t,\tau_2)}{\partial_2 G(0,\tau_2)} - (1 - H_t^2)\frac{\partial_1 G(t,t)}{G(0,t)}\right) dt$$

where $M_t^2 = H_t^2 - \int_0^{t\wedge\tau_2} \frac{\partial_2 G(0,s)}{G(0,s)} ds$ is a $\mathbb{H}^2$-martingale.

Valuation of a Defaultable claim

Let us now examine the valuation of a simple defaultable claim which delivers $\delta(\tau_1)$ at time τ_1, if $\tau_1 < T$, where δ is a deterministic function. We assume zero interest rate and that the pricing is done under $\mathbb{P}$.

The value S of this claim, computed in the filtration $\mathbb{H}$, i.e., taking care on the information on the second default contained in that filtration, is

$$S_t = \mathbb{1}_{\{t<\tau_1\}} \mathbb{E}\left(\delta(\tau_1) \mathbb{1}_{\tau_1 \leq T} | \mathcal{H}_t\right)$$

Let us denote by $\tau = \tau_1 \wedge \tau_2$ the moment of the first default. Then, $1\!\!1_{\{t<\tau\}} S_t = 1\!\!1_{\{t<\tau\}} \widetilde{S}_t$, where

$$\widetilde{S}_t = \frac{1}{G(t,t)} \mathbb{E}(\delta(\tau_1) 1\!\!1_{\tau_1 \leq T})$$

$$\widetilde{S}_t = \frac{1}{G(t,t)} \left(- \int_t^T \delta(u) \partial_1 G(u,t)\, du \right)$$

where $G(t,t) = \mathbb{P}(\tau > t)$.

Hence the dynamics of the *pre-default price* $\widetilde{S}_t$ are

$$d\widetilde{S}_t = \Big(\big(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)\big)\widetilde{S}_t - \widetilde{\lambda}_1(t)\delta(t) - \widetilde{\lambda}_2(t)S_t^{1|2}(t)\Big)\, dt,$$

where for $i = 1, 2$ the function $\widetilde{\lambda}_i(t)$ is the (deterministic) *pre-default intensity* of τ_i and $S_t^{1|2}(t)$ is given by the expression

$$S_t^{1|2}(t) = \frac{1}{\partial_2 G(t,t)} \left(- \int_t^T \delta(u) f(u,t)\, du \right).$$

In the financial interpretation, $S_t^{1|2}(t)$ is the price at time t of the claim on the first credit name, under the assumption that the default τ_2 occurs at time t and the first name has not yet defaulted (recall that simultaneous defaults are excluded).

Let us now consider the event $\{\tau_2 \leq t < \tau_1\}$. The price of the claim equals

$$S_t^{1|2}(\tau_2) = \frac{1}{\partial_2 G(t,\tau_2)} \left(- \int_t^T \delta(u) f(u,\tau_2)\, du \right).$$

Consequently, on the event $\{\tau_2 \leq t < \tau_1\}$ we obtain

$$dS_t = \left(\lambda^{1|2}(t,\tau_2)\, (S_t - \delta(t)) \right) dt$$

It follows that

$$
\begin{aligned}
dS_t &= -(1-H_t^1)(1-H_t^2)\delta(t)\widetilde{\lambda}^1(t))dt - (1-H_t^1)H_t^2\delta(t)\widetilde{\lambda}_t^{1|2}dt \\
&\quad -S_{t-}dM_t^1 + (1-H_t^1)(S_t^{1|2}(t) - S_{t-})dM_t^2 \\
&= (\delta(t) - S_{t-})dM_t^1 + (1-H_t^1)(S_t^{1|2}(t) - S_{t-})dM_t^2 - \delta(t)dH_t^1
\end{aligned}
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Example: Jarrow and Yu's Model

Let $\tau_i = \inf\{t : \Lambda_i(t) \geq \Theta_i\}, i = 1, 2$ where $\Lambda_i(t) = \int_0^t \lambda_i(s)ds$ and Θ_i are independent random variables with exponential law of parameter 1. Jarrow and Yu study the case where λ_1 is a constant and

$$\lambda_2(t) = \lambda_2 1\!\!1_{\{t<\tau_1\}} + \alpha_2 1\!\!1_{\{\tau_1 \leq t\}} .$$

Assume for simplicity that $r = 0$ and compute the value of a defaultable zero-coupon with default time τ_i, with a rebate δ_i:

$$D_i(t, T) = E(1\!\!1_{\{\tau_i > T\}} + \delta_i 1\!\!1_{\{\tau_i < T\}} | \mathcal{H}_t), \text{ for } \mathcal{H}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2 .$$

After some computation

$$
\begin{aligned}
D_1(t,T) &= \delta_1 + 1\!\!1_{\{\tau_1>t\}}(1-\delta_1)e^{-\lambda_1(T-t)} \\
D_2(t,T) &= \delta_2 + (1-\delta_2)1\!\!1_{\{\tau_2>t\}}\Big(1\!\!1_{\{\tau_1\le t\}}e^{-\alpha_2(T-t)} \\
&\quad + 1\!\!1_{\{\tau_1>t\}}\frac{1}{\lambda_1+\lambda_2-\alpha_2}(\lambda_1 e^{-\alpha_2(T-t)} + (\lambda_2-\alpha_2)e^{-(\lambda_1+\lambda_2)(T-t)})\Big)
\end{aligned}
$$

Copula-Based Approaches

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(ii) $C(u_1,\dots,u_n)$ is increasing with respect to each component u_i

(iii) For any $a, b \in [0,1]^n$ with $a \le b$ (i.e., $a_i \le b_i$, $\forall i$)

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(u_{1,i_1},\dots,u_{n,i_n}) \ge 0\,,$$

where $u_{j,1} = a_j$, $u_{j,2} = b_j$.

Let us give few examples of copulas:

- Product copula: $\Pi(u_1, \ldots, u_n) = \Pi_{i=1}^n u_i$,
- Gumbel copula: for $\theta \in [1, \infty)$ we set

$$C(u_1, \ldots, u_n) = \exp\left(- \left[\sum_{i=1}^n (-\ln u_i)^\theta \right]^{1/\theta} \right),$$

- Gaussian copula:

$$C(u_1, \ldots, u_n) = N_\Sigma^n \left(N^{-1}(u_1), \ldots, N^{-1}(u_n) \right),$$

where N_Σ^n is the c.d.f for the n-variate central normal distribution with the linear correlation matrix Σ, and N^{-1} is the inverse of the c.d.f. for the univariate standard normal distribution.

Sklar Theorem:

For any cumulative distribution function F on $\mathbb{R}^n$ there exists a copula function C such that

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

where F_i is the i^{th} marginal cumulative distribution function. If, in addition, F is continuous then C is unique.

Direct Application

Let F_i be the probability distribution for τ_i. A copula function C is chosen in order to introduce a dependence structure of the random vector $(\tau_1, \tau_2, \ldots, \tau_n)$. The joint distribution of the random vector $(\tau_1, \tau_2, \ldots, \tau_n)$ is derived by

$$\mathbb{P}\{\tau_i \leq t_i,\ i = 1, 2, \ldots, n\} = C\big(F_1(t_1), \ldots, F_n(t_n)\big).$$

Indirect Application

Assume that the cumulative distribution function of $(\xi_1, \ldots, \xi_n)$ is given by an n-dimensional copula C, and that the univariate marginal laws are uniform on $[0, 1]$.

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Assume that the cumulative distribution function of $(\xi_1, \ldots, \xi_n)$ is given by an n-dimensional copula C, and that the univariate marginal laws are uniform on $[0, 1]$. We postulate that $(\xi_1, \ldots, \xi_n)$ are independent of $\mathbb{F}$, and we set

$$\tau_i = \inf \{ t \, : \, \Lambda_t^i \geq - \ln \xi_i \}.$$

Then:

- The case of default times conditionally independent with respect to $\mathbb{F}$ corresponds to the choice of the product copula Π. In this case, for $t_1, \ldots, t_n \leq T$ we have

$$\mathbb{P}\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = \Pi(Z^1_{t_1}, \ldots, Z^n_{t_n}),$$

  where we set $Z^i_t = e^{-\Lambda^i_t}$.

- In general, for $t_1, \ldots, t_n \leq T$ we obtain

$$\mathbb{P}\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = C(Z^1_{t_1}, \ldots, Z^n_{t_n}),$$

  where C is the copula used in the construction of $\xi_1, \ldots, \xi_n$.

An example

This example describes the use of one-factor Gaussian copula (Bank of International Settlements (BIS) standard).

Let q_i be a decreasing function taking values in $[0,1]$ with $q_i(0)=1$.

$$\tau_i = \inf\{t : q_i(t) < U_i\}$$

Then, $q_i(t) = \mathbb{P}(\tau_i > t) = 1 - p_i(t)$.

Correlation specification of the thresholds U_i: Let $Y_1, \cdots, Y_n$ and Y be independent random variables and $X_i = \rho_i Y + \sqrt{1-\rho_i^2} Y_i$.

The default thresholds are defined by $U_i = 1 - F_i(X_i)$ where F_i is the cumulative (continuous) distribution function of X_i. Then

$$\tau_i = \inf\{t : \rho_i Y + \sqrt{1-\rho_i^2} Y_i \le F_i^{-1}(1 - q_i(t))\}.$$

Conditioned on the common factor Y,

$$p^i(t|Y) = F_i^Y\left(\frac{F_i^{-1}(p_i(t)) - \rho_i Y}{\sqrt{1-\rho_i^2}}\right)$$

where F_i^Y is the cumulative distribution function of Y_i.

Let us consider the particular case where

$$X_i = \rho_i Y + \sqrt{1-\rho_i^2} Y_i,$$

where Y, Y_i, $i = 1, 2, \ldots, n$, are independent standard Gaussian variables. In that case, X_i is also a standard Gaussian law and

$$p^i(t|Y) = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(p_i(t)) - \rho_i Y}{\sqrt{1-\rho_i^2}}\right)$$

and

$$\mathbb{P}(\tau_i \leq t_i, \forall i \leq n) = \int \prod_i \mathcal{N}\left(\frac{\mathcal{N}^{-1}(F_i(t_i)) - \rho_i y}{\sqrt{1-\rho_i^2}}\right) f(y) dy\,.$$

where f is the density of Y

Loss

The cumulative loss on the underlying portfolio is $L_t = \sum_{i=1}^{n} N_i(1 - R_i)\mathbb{1}_{\tau_i \leq t}$ where N_i is the nominal value of each firm and R_i is the (constant) recovery rate. The first defaults only affect the equity tranche until the cumulative loss has arrived the total nominal amount of the equity tranche and the loss on the tranche is given by

$$L_t^E = L_t \mathbb{1}_{0,n^E}(L_t) + n^E \mathbb{1}_{n^E,\infty}(L_t) = L_t - (L_t - n^E)^+$$

Conditional on the common factor Y , we can rewrite

$$L_T = \sum N_i(1 - R_i)\mathbb{1}_{Y_i \leq \,(\,F_i^{-1}(p_i(T)) - \rho_i Y\,)\,/\,(1 - \rho_i^2)}$$

Hence, the conditional total loss L_T w.r.t. the factor Y can be written as the sum of independent Bernoulli random variables, each with probability $p_i(T|Y)$

Survival Intensities

For arbitrary $s \leq t$ on the set $\{\tau_1 > s, \ldots, \tau_n > s\} = \{\tau_{(1)} > s\}$ we have

$$\mathbb{P}\{\tau_i > t \,|\, \mathcal{G}_s\} = \mathbb{E}_{\mathbb{P}}\left(\frac{C(Z_s^1, \ldots, Z_t^i, \ldots, Z_s^n)}{C(Z_s^1, \ldots, Z_s^n)} \,\Big|\, \mathcal{F}_s \right).$$

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PROOF: The proof is straightforward, and follows from the key lemma

$$\mathbb{P}\{\tau_i > t \,|\, \mathcal{G}_s\} \mathbb{1}_{\{\tau_{(1)} > s\}} = \mathbb{1}_{\{\tau_{(1)} > s\}} \frac{\mathbb{P}(\tau_1 > s, \ldots, \tau_i > t, \ldots, \tau_n > s \,|\, \mathcal{F}_s)}{\mathbb{P}(\tau_1 > s, \ldots, \tau_i > s, \ldots, \tau_n > s \,|\, \mathcal{F}_s)}$$

$\triangle$

Consequently, assuming that the derivatives $\gamma_t^i = \frac{d\Lambda_t^i}{dt}$ exist, the i-th intensity of survival equals, on the set $\{\tau_1 > t, \dots, \tau_n > t\}$,

$$\lambda_t^i = \gamma_t^i \, Z_t^i \, \frac{\frac{\partial}{\partial v_i} C(Z_t^1, \dots, Z_t^n)}{C(Z_t^1, \dots, Z_t^n)} = \gamma_t^i \, Z_t^i \, \frac{\partial}{\partial v_i} \ln C(Z_t^1, \dots, Z_t^n),$$

where λ_t^i is understood as the limit:

$$\lambda_t^i = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}\{t < \tau_i \leq t + h \,|\, \mathcal{F}_t, \tau_1 > t, \dots, \tau_n > t\}.$$

It appears that, in general, the i-th intensity of survival jumps at time t, if the j-th entity defaults at time t for some $j \neq i$. In fact, it holds that

$$\lambda_t^{i,j} = \gamma_t^i \, Z_t^i \, \frac{\frac{\partial^2}{\partial v_i \partial v_j} \, C(Z_t^1, \ldots, Z_t^n)}{\frac{\partial}{\partial v_j} \, C(Z_t^1, \ldots, Z_t^n)} \, ,$$

where

$$\lambda_t^{i,j} = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}\{t < \tau_i \leq t + h \,|\, \mathcal{F}_t, \tau_k > t, k \neq j, \tau_j = t\}.$$

Schönbucher and Schubert (2001) also examine the intensities of survival after the default times of some entities. Let us fix s, and let $t_i \leq s$ for $i = 1, 2, \ldots, k < n$, and $T_i \geq s$ for $i = k+1, k+2, \ldots, n$. Then,

$$\mathbb{Q}\Big\{\tau_i > T_i,\ i = k+1, k+2, \ldots, n \,|\, \mathcal{F}_s,\ \tau_j = t_j,\ j = 1, 2, \ldots, k,$$

$$\tau_i > s,\ i = k+1, k+2, \ldots, n\Big\}$$

$$= \frac{\mathbb{E}_{\mathbb{Q}}\left(\frac{\partial^k}{\partial v_1 \ldots \partial v_k} C(Z^1_{t_1}, \ldots, Z^k_{t_k}, Z^{k+1}_{T_{k+1}}, \ldots, Z^n_{T_n}) \,\Big|\, \mathcal{F}_s \right)}{\frac{\partial^k}{\partial v_1 \ldots \partial v_k} C(Z^1_{t_1}, \ldots, Z^k_{t_k}, Z^{k+1}_s, \ldots, Z^n_s)}.$$

Density models

We introduce the **conditional joint survival process** $G_t(u,v)$

$$G_t(u,v) = \mathbb{Q}(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t).$$

We write

$$\partial_1 G_t(u,v) = \frac{\partial}{\partial u} G_t(u,v), \quad \partial_{12} G_t(u,v) = \frac{\partial^2}{\partial u \partial v} G_t(u,v)$$

so that

$$\boxed{G_t(u,v) = \int_u^\infty dx \int_v^\infty g_t(x,y)\, dy}$$

where $(g_t(x,y), t \geq 0)$ is a family of $\mathbb{F}$-predictable processes (in fact $(\mathbb{F},\mathbb{Q})$-martingales) and, if $\mathbb{F}$ is a Brownian filtration

$$G_t(u,v) = G_0(u,v) + \int_0^t \sigma_s(u,v) dW_s$$

An example (Fermanian, Crépey)

Define $\tau_i = h_i(\int_0^\infty f(s)dB^i_s)$ where B^i are BM, with the same correlation coefficient, f is a deterministic function and h_i are positive functions, increasing. Then, the conditional joint law of $(\tau_i, i \leq)$ correspond to a Gaussian copula

Construction

To produce such modeling, one can work as follows

• Start with the case where $\tau = (\tau_1, \ldots, \tau_n)$ is independent from $\mathbb{F}$ and note η the law of τ. Work, on the canonical product space, under $\mathbb{P}^0 = \mathbb{P} \times \eta$

• Define the counting process $N_t = \sum_n 1\!\!1_{\tau_n \leq t}$ and its natural filtration $(\mathcal{N}_t, t \geq 0)$. Compute, in that "toy model" the conditional law of τ given $\mathcal{N}_t$

• Do a change of probability $d\mathbb{Q} = \beta_T d\mathbb{P}^0$ on the product space. Use Bayes formula to compute various conditional expectation

This construction extends easily to the case where some information on marks associated with τ are given.

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$(T:x)$ bonds

Filipovic et al. consider the loss process L, defined as a increasing marked point process μ, valued in $[0,1]$ with an absolutely continuous compensator $\nu(t,dx)dt$. They introduce $(T:x)$ bonds, with price $P(t,x,T)$ which have a payoff at time T equal to $\mathbb{1}_{L_T\leq x}$, and the $(T:x)$ forward rate f so that

$$P(t,x,T)=\mathbb{1}_{L_t\leq x}\exp(-\int_t^T f(t,x,u)du)$$

They prove that the compensator of $\mathbb{1}_{L_t\leq x}$ is $\int_0^t \mathbb{1}_{L_s\leq x}\lambda(s,x)ds$, where $\lambda(t,x)=\nu(t,(x-L_t,1]\cap[0,1])$. Assuming that

$$d_t f(t,x,T)=a(t,x,T)dt+b(t,x,T)dW_t+\int c(t,x,T;y)\mu(dt,dy)$$

they show that the no-arbitrage condition is

$$\begin{aligned}\int_t^T a(t,x,u)du &= \frac{1}{2}(\int_t^T b(t,x,u)du)^2+\int(e^{-\int_t^T c(t,x,u;y)du-1}-1)\mathbb{1}_{L_t+y\leq x}\nu(t,dy)\\ r_t+\lambda(t,x) &= f(t,x,t)\end{aligned}$$

Markov Chain based models

Let $\mathrm{H} = (H^1, H^2)$ denote the pair of the default indicator processes, so $H^i_t = \mathbb{1}_{\tau_i \le t}$.

Given a *factor process* $\mathrm{X} = (X_1, X_2)$, we consider a Markovian model of the pair (X, H), with generator given by, for $u = u(t, x, e)$ with $t \in \mathbb{R}_+, x = (x_1, x_2) \in \mathbb{R}^2, e = (e_1, e_2) \in \{0, 1\}^2$:

$$\begin{aligned}\mathcal{A}u(t,x,e) &= \partial_t u(t,x,e)\\ &+ \sum_{1\le i\le 2} \ell_i(t,x_i)\left(u(t,x,e^i) - u(t,x,e)\right) + \ell_3(t)\left(u(t,x,1,1) - u(t,x,e)\right)\\ &+ \sum_{1\le i\le 2} \left(b_i(t,x_i)\partial_{x_i} u(t,x,e) + \frac{1}{2}\sigma_i^2(t,x_i)\partial^2_{x_i^2} u(t,x,e)\right),\end{aligned}$$

where, for $i = 1, 2$:

- e^i is the vector obtained from e, by replacing the component i by number one,
- b_i and σ_i^2 denote factor *drift* and *variance* functions, and ℓ_i is an *individual default intensity* function, and $\ell_3(t)$ stands for the *joint defaults intensity* function.

The $\mathbb{F}$-*intensity-matrix function* of H is thus given by the following 4×4 matrix $A(t,x)$, where the first to fourth rows (or columns) correspond to the four possible states $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ of H_t :

$$A(t,x) = \begin{bmatrix} -\ell(t,x) & \ell_1(t,x_1) & \ell_2(t,x_2) & \ell_3(t) \\ 0 & -q_2(t,x_2) & 0 & q_2(t,x_2) \\ 0 & 0 & -q_1(t,x_1) & q_1(t,x_1) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with, for every $i = 1, 2$,

$$q_i(t,x_i) = \ell_i(t,x_i) + \ell_3(t)$$

and $\ell(t,x) = \ell_1(t,x_1) + \ell_2(t,x_2) + \ell_3(t)$.

For every $i = 1, 2$, the process (X^i, H^i) is an $\mathbb{F}$-Markov process with generator given by, for $u = u(t, x, e)$ with $t \in \mathbb{R}_+, x \in \mathbb{R}, e \in \{0, 1\}$:

$$\begin{aligned}\mathcal{A}_i u(t,x,e) =& \partial_t u(t,x,e) + b_i(t,x)\partial_x u(t,x,e) + \frac{1}{2}\sigma_i^2(t,x)\partial^2_{x^2} u(t,x,e) \\ &+ q_i(t,x)\left(u(t,x,1) - u_i(t,x,e)\right) \ .\end{aligned}$$

The $\mathbb{F}$-intensity matrix function of H^i is thus given by

$$A_i(t,x) = \begin{bmatrix} -q_i(t,x) & q_i(t,x) \\ 0 & 0 \end{bmatrix}$$

In other words, the process M^i defined by, for $i = 1, 2$,

$$M^i_t = H^i_t - \int_0^t (1 - H^i_s) q_i(s, X^i_s) ds \ ,$$

is an $\mathbb{F}$-martingale.

One has, for every $t \geq 0$,

$$\mathbb{P}(\tau_i > t) = \mathbb{E}\exp\left(-\int_0^t q_i(u, X_u^i)du\right) \quad , \quad \mathbb{P}(\tau_1 \wedge \tau_2 > t) = \mathbb{E}\exp\left(-\int_0^t \ell(u, \mathrm{X}_u)du\right) .$$

Particular case

We model the pair $H = (H^1, H^2)$ as an inhomogeneous Markov chain with state space $E = \{(0,0), (1,0), (0,1), (1,1)\}$, and generator matrix at time t given by the following matrix $A(t)$, where the first to fourth rows (or columns) correspond to the four possible states $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ of H_t :

$$A(t) = \begin{bmatrix} -(\ell_1(t) + \ell_2(t) + \ell_3(t)) & \ell_1(t) & \ell_2(t) & \ell_3(t) \\ 0 & -(\ell_2(t) + \ell_3(t)) & 0 & \ell_2(t) + \ell_3(t) \\ 0 & 0 & -(\ell_1(t) + \ell_3(t)) & \ell_1(t) + \ell_3(t) \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

where ℓ's are deterministic functions of time.

Let us further introduce the processes $H^{\{1\}}$, $H^{\{2\}}$ and $H^{\{1,2\}}$ standing for the indicator processes of a default of the firm alone, of the counterpart alone, and of a simultaneous default of the firm and the counterpart, respectively.

$$H_t^{\{1\}} = 1\!\!1_{\tau_1 \leq t, \tau_1 \neq \tau_2} \ , \ H_t^{\{2\}} = 1\!\!1_{\tau_2 \leq t, \tau_1 \neq \tau_2} \ , \ H_t^{\{1,2\}} = 1\!\!1_{\tau_1 = \tau_2 \leq t} \ .$$

The $\mathbb{H}$-intensity of H^ι is of the form $q_\iota(t,H_t) = q_\iota(t,H_t^1,H_t^2)$ for a suitable function $q_\iota(t,h)$:

$$q_{\{1\}}(t,h) = 1\!\!1_{h_1=0}\left(1\!\!1_{h_2=0}\ell_1(t) + 1\!\!1_{h_2=1}(\ell_1+\ell_3)(t)\right)$$

$$q_{\{2\}}(t,h) = 1\!\!1_{h_2=0}\left(1\!\!1_{h_1=0}\ell_2(t) + 1\!\!1_{e_1=1}(\ell_2+\ell_3)(t)\right)$$

$$q_{\{1,2\}}(t,h) = 1\!\!1_{h=(0,0)}\ell_3(t)\ .$$

The processes M^i defined by, for $i=1,2$,

$$M_t^i = H_t^i - \int_0^t (1-H_s^i)(\ell_i+\ell_3)(s)ds\ ,$$

are $\mathbb{H}$-martingales.

The processes H^1 and H^2 are $\mathbb{H}$-Markov processes

One has,

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \exp\left(-\int_0^s \ell_1(u)du - \int_0^t \ell_2(u)du - \int_0^{s\vee t}\ell_3(u)du\right)$$

Self-exciting Models, Multiphase models

Self-exciting Models

A basic example is a Hawkes process, which is specified by a positive functions λ_k. The intensity of the counting process $N_t = \sum \mathbb{1}_{\tau_k \leq t}$ is
$\lambda_t = \lambda_0(t) + \sum_{k, \tau_k < t} \lambda_k(t - \tau_k)$

In this specification, the intensity of N is updated with default information along the path. The construction of N can be done using change of time procedure.

Generalisation:

$$
\begin{aligned}
L_t &= \sum \mathbb{1}_{\tau_k < t} \\
\tau_k &= \inf\{t \, : \int_0^t \lambda_s^k \geq \Theta_k\} \\
d\lambda_t^k &= -\alpha_k(\lambda_t^k - \widehat{\lambda}_k)dt + \sigma_k \sqrt{\lambda_t^k} dW_t + b_k dL_t + c_k \lambda_t^k dX_t
\end{aligned}
$$

Multiphase models

Consider a continuous-time Markov chain X_t is on a state space $\boldsymbol{E} = 1, 2, \ldots, m, \Delta$ where the states $1, 2, \ldots, m$, are transient and Δ is an absorbing state. The generator $\boldsymbol{\Lambda}$ of the chain X_t is given by

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{A}\boldsymbol{e} \\ \\ \mathbf{0} & 0 \end{bmatrix}$$

where $\boldsymbol{e}$ is a column vector in $\mathbb{R}^m$ where all entries equals 1.

Let Γ_1 and Γ_2 be two stochastically closed subsets of $\boldsymbol{E}$, which means that once X_t enters Γ_i it never leaves Γ_i. We assume that $\Gamma_1 \cap \Gamma_2$ is reduced to Δ.

Then, the matrix $\boldsymbol{A}$ can be formulated as an upper diagonal matrix, i.e. the elements below the diagonal is zero.

Let τ_i be defined as

$$\tau_i = \inf\{t > 0 : X_t \in \Gamma_i\}, \qquad i = 1, 2.$$

and $\boldsymbol{g}_i$ be a $m \times m$ diagonal matrices where $(\boldsymbol{g}_i)_{k,k} = \mathbb{1}_{\{k \in \Gamma_i^c\}}$, that is, the k-th diagonal of $\boldsymbol{g}_i$ for $k = 1, 2, \ldots, m$ equals 1 if $k \in \Gamma_i^c$ and zero otherwise.

For example, in the case of the two dimensional states (X_1, X_2) with four states $(0,0), (0,1), (1,0), (1,1)$ where 1 is the cemetery, we have $m = 3$, and

$$g_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad g_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, for an initial distribution given by $\boldsymbol{\alpha}$,

$$\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2) = \begin{cases} \boldsymbol{\alpha} e^{\boldsymbol{A} t_2} \boldsymbol{g}_2 e^{\boldsymbol{A}(t_1 - t_2)} \boldsymbol{g}_1 \boldsymbol{e} & \text{if } \ 0 \le t_2 \le t_1, \\ \boldsymbol{\alpha} e^{\boldsymbol{A} t_1} \boldsymbol{g}_1 e^{\boldsymbol{A}(t_2 - t_1)} \boldsymbol{g}_2 \boldsymbol{e} & \text{if } \ 0 \le t_1 \le t_2 \end{cases}$$

The absolute continuous component of the joint law has density $f(t_1, t_2)$ given by

$$f(t_1,t_2)=\begin{cases}\boldsymbol{\alpha} e^{\boldsymbol{A}t_2}\left[\boldsymbol{A},\boldsymbol{g}_2\right]e^{\boldsymbol{A}(t_1-t_2)}\boldsymbol{A}\boldsymbol{g}_1\boldsymbol{e} & \text{if } \ 0\le t_2\le t_1,\\ \boldsymbol{\alpha} e^{\boldsymbol{A}t_1}\left[\boldsymbol{A},\boldsymbol{g}_1\right]e^{\boldsymbol{A}(t_2-t_1)}\boldsymbol{A}\boldsymbol{g}_2\boldsymbol{e} & \text{if } \ 0\le t_1\le t_2\end{cases}$$

where $[\boldsymbol{A},\boldsymbol{B}]=\boldsymbol{A}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{A}$.

Furthermore

$$\mathbb{P}(\tau_1=\tau_2>t)=\boldsymbol{\alpha} e^{\boldsymbol{A}t}\boldsymbol{A}^{-1}\left(\boldsymbol{A}\boldsymbol{g}_1\boldsymbol{g}_2-\left[\boldsymbol{A},\boldsymbol{g}_1\right]\boldsymbol{g}_2-\left[\boldsymbol{A},\boldsymbol{g}_2\right]\boldsymbol{g}_1\right)\boldsymbol{e}.$$

Partial Observation, Filtering Problems

Since the seminal paper of Duffie and Lando, a partial observation methodology allows to built a bridge between structural approach and reduced form approach.

The first application is to assume that

$$\tau = \inf\{t : X_t \leq 0\}$$

where X is some driving process with natural filtration $\mathbb{F}^X$, and the observation $\mathbb{F}$ is a filtration smaller than $\mathbb{F}^X$. For example, one observes the process X only at discrete times (t_i), which may be random. One can also assume that one observes the process with some noise.

Filtering problems in credit risk were introduced by Nakagawa.

Other models assume that the default indicators are on the form

$$dH_t^i = (1 - H_{t-}^i) \int \kappa_i(t, X_{t-}, u) \mathcal{N}(du, dt)$$

where κ takes values in $\{0, 1\}$ and $\mathcal{N}$ is a Poisson random measure, and X is a jump-diffusion (or a Markov Chain)

The filtration of observation is then the one given by Y and some "observation process" Z following

$$dZ_t = a(t, X_t)dt + dB_t$$